The Virial Theorem for Stars

Stars are excellent examples of systems in virial equilibrium. To see this, let us make two assumptions:

1) Stars are in hydrostatic equilibrium
2) Stars are made up of ideal gases.

Now consider the total internal energy of a star

$$E_i = \int_0^{\mathcal{M}_T} u \, d\mathcal{M} = \int_0^{\mathcal{M}_T} \frac{3}{2} \frac{N_A}{\mu} k \, T \, d\mathcal{M} \tag{10.1.1}$$

If we substitute in the ideal gas law

$$P = \frac{\rho N_A}{\mu} k \, T \tag{10.1.2}$$

then

$$E_i = \int_0^{\mathcal{M}_T} \frac{3}{2} \frac{P}{\rho} \, d\mathcal{M} \tag{10.1.3}$$

Now take the equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{G\mathcal{M}(r)}{r^2} \rho \tag{2.2.2}$$

If we multiply this equation by $4\pi r^3 dr$, and integrate over the entire star, we get

$$\int_0^R \frac{dP}{dr} \cdot 4\pi r^3 dr = -\int_0^R \frac{G\mathcal{M}(r)}{r} \rho \cdot 4\pi r^2 dr$$

When integrated by parts, the left side of the equation is

$$\int_0^R \frac{dP}{dr} \cdot 4\pi r^3 dr = P(r) \, 4\pi r^3 \Big]_0^R - \int_0^R 12\pi r^2 P \, dr$$

so

$$P(R)\,4\pi R^3 - \int_0^R 12\pi r^2 P\,dr = -\int_0^R \frac{G\mathcal{M}(r)}{r}\rho\cdot 4\pi r^2 dr \quad (10.1.4)$$

Now note that the first term on the left side of the equation goes away, since the pressure at the surface is $P(R) = 0$. Note also, that the right side of equation is just the total gravitational energy of the entire star. Thus,

$$\int_0^R 12\pi r^2 P dr = 2\int_0^R \frac{3}{2} P \cdot 4\pi r^2 dr = -E_{\rm grav} \qquad (10.1.5)$$

or, since $d\mathcal{M} = 4\pi r^2 \rho\, dr$,

$$2\int_0^{\mathcal{M}_T} \frac{3}{2}\,\frac{P}{\rho} d\mathcal{M} = E_{\rm grav}$$

This is simply virial equilibrium

$$2\,E_i + E_{\rm grav} = 0 \qquad (10.1.6)$$

The expression is independent of $\mathcal{M}(r)$, $\rho(r)$, and $T(r)$, and can even be generalized for other equations of state. For ideal gases

$$P = \frac{\rho N_A}{\mu} k\,T = \frac{2}{3}\,\rho\,u$$

but more generally, we can write

$$P = (\gamma - 1)\,\rho\,u \qquad (10.1.7)$$

where $\gamma = 5/3$ for an ideal gas. Under this law, (10.1.3) becomes

$$E_i = \int_0^{\mathcal{M}_T} \frac{1}{\gamma - 1}\,\frac{P}{\rho}\,d\mathcal{M}$$

the equation of virial equilibrium becomes

$$3(\gamma - 1)E_i + E_{\rm grav} = 0 \tag{10.1.8}$$

and the total energy of the star, $W = E_i + E_{\rm grav}$, is

$$W = -\frac{E_{\rm grav}}{3(\gamma - 1)} + E_{\rm grav} = \frac{3\gamma - 4}{3\gamma - 3} E_{\rm grav} \tag{10.1.9}$$

Note the implication of these equations. Because stars have non-zero temperatures, they will radiate some of their energy into space. Thus, through energy conservation

$$\mathcal{L}_T + \frac{dW}{dt} = 0$$

or

$$\mathcal{L}_T = -\frac{3\gamma - 4}{3\gamma - 3} \, \frac{dE_{\rm grav}}{dt} \tag{10.1.10}$$

This means that as a star radiates, its gravitational energy will become more negative, but this decrease will be **less** than the amount of energy radiated. The remaining energy will go into heating the star. In the case of an ideal gas, one half the radiated energy will go into $E_{\rm grav}$, and the other half into E_i.

Note also from (10.1.9), that since $E_{\rm grav} < 0$, the total energy of a star will be negative (which means the star is bound). Only if the star is completely relativistically degenerate ($\gamma = 4/3$) is $W = 0$.

Stellar Timescales

There are several characteristic timescales associated with the evolution of stars. These timescales are extremely important, in that they allow us to make quick order of magnitude calculations about the importance of various physical processes.

THE KELVIN-HELMHOLTZ TIME SCALE

The characteristic timescale for energy release from gravitational contraction can be computed simply from the amount of energy available, $E_{\rm grav}$ and the rate of energy loss, $\mathcal{L}$. From (10.1.4)

$$E_{\rm grav} = \int_0^R \frac{G\mathcal{M}(r)}{r}\,\rho\cdot 4\pi r^2\,dr = \int_0^{\mathcal{M}_T} \frac{G\mathcal{M}(r)}{r}\,d\mathcal{M} = q\,\frac{G\mathcal{M}_T^2}{R}$$

where q is a number of the order unity. The Kelvin-Helmholtz timescale is then

$$\tau_{\rm KH} = \frac{G\mathcal{M}^2}{R\mathcal{L}} \tag{10.2.1}$$

The Kelvin-Helmholtz timescale is also sometimes called the thermal timescale; it gives a rough idea of how long a star can shine without internal sources of energy. Similarly, it also describes how long it takes for thermal energy produced in the center of the star via gas pressure, to work its way out via energy transport.

THE FREE-FALL TIME SCALE

The timescale for a star's core to feel gravitational changes in surface of the star is the dynamical, or free-fall timescale. Consider an impact at the stellar surface. A pressure wave caused by this impact will reach the center of the star in

$$\tau_{\rm ff} \sim \frac{R}{v_s} = \frac{R}{\sqrt{\gamma P/\rho}} \sim \frac{R}{\sqrt{P/\rho}}$$

where v_s is the speed of sound. If the star is near hydrostatic equilibrium, then

$$\frac{G\mathcal{M}}{r^2}\rho = -\frac{dP}{dr} \sim -\frac{P_{\rm cen} - P_{\rm sur}}{r_{\rm cen} - r_{\rm sur}} \sim \frac{P}{R}$$

which gives

$$\frac{P}{\rho} \sim \frac{G\mathcal{M}}{R}$$

and

$$\tau_{\rm ff} \sim \left(\frac{R^3}{G\mathcal{M}}\right)^{1/2} \tag{10.2.2}$$

This timescale describes how quickly the star mechanically adjusts to changes.

THE NUCLEAR TIME SCALE

Another useful timescale to know is how long it takes the reservoir of nuclear energy in the star to be released, *i.e.*,

$$\tau_{\rm nuc} = \frac{E_{\rm nuc}}{\mathcal{L}} = \frac{Q\mathcal{M}}{\mathcal{L}} \tag{10.2.3}$$

where Q is the energy released per unit gram for the nuclear reaction being considered. For hydrogen burning, $Q = 6.3 \times 10^{18}$ ergs-g^{-1}, while for helium burning, it is a factor of ~ 10 smaller.

Chemical Composition of Stars

For abundance studies in the ISM, one normally quotes n_i, the number density of particles of species i per unit volume. For stellar modeling, however, the variable X_i is used, which is the mass fraction of species i. X_i and n_i are related by

$$X_i = \frac{n_i A_i}{N_A \rho} \tag{10.3.1}$$

where A_i is the species' atomic weight and N_A is Avagadro's number. With this definition, the abundances are normalized, such that

$$\sum_i X_i = 1 \tag{10.3.2}$$

A star is normally assumed to be chemically homogeneous when it starts its life, but nuclear reactions will quickly change its composition. If r_{ij} is the number of reactions per unit volume that change species i into species j, then clearly, the change in the abundance of element i with time will be

$$\frac{dX_i}{dt} = \frac{A_i}{\rho N_A} \left[\sum_j r_{ji} - \sum_k r_{ik} \right] \tag{10.3.3}$$

(We will consider the reaction rates, r_{ji} in the next lecture.) The first term in the bracket represents the creation of element i, while the second term gives the destruction of element i. Associated with each of these reactions is an energy (per unit mass), ϵ_{ij}. If Q_{ij} is the amount of energy generated (or lost) when one particle of i reacts to form j, then

$$\epsilon_{ij} = \frac{r_{ij} Q_{ij}}{\rho} \tag{10.3.4}$$

Stars can become extremely inhomogeneous as a result of nuclear reactions. However, in regions where convection occurs, this inhomogeneity immediately disappears. (The timescale for convection is extremely short.) Thus, within a convective region, $\frac{dX_i}{dm} = 0$.

Nuclear Reaction Energies

To compute the amount of energy generated by nuclear reactions, we first must compute the energy associated with a single reaction. To do this, consider the following reaction between particle or photon a and a nucleus X

$$a + X \longrightarrow Y + b \tag{10.4.1}$$

which, in shorthand notation, is expressed as $X(a,b)Y$. Naturally, during this process, charge, nucleon number, momentum, and energy are all conserved. So

$$E_{aX} + (M_a + M_X)\,c^2 = E_{bY} + (M_b + M_Y)\,c^2$$

Since nucleon number is conserved, this equation can be rewritten by subtracting the nucleon number (in a.m.u.'s) from each side

$$E_{aX} + \Delta M_a + \Delta M_X = E_{bY} + \Delta M_b + \Delta M_Y \tag{10.4.2}$$

where ΔM is the atomic mass excess of each particle in units of energy. By definition, $\Delta M_{C^{12}} = 0$. For reference, H^1 has a mass excess of $+7.289$ MeV, while $\Delta M_{Fe^{56}} = -60.605$ MeV. (Actually, the quoted values for mass excess are usually off by a couple of eV, since the tables are based on atomic masses, not nuclear masses, and different elements have different electron binding energies. Compared to the nuclear binding energies, this error is negligible.)

Note that the above definition of Q assumes all of the mass defect goes into usable energy for the star. However, if the reaction involves the weak nuclear force, neutrinos will also be produced. Neutrino energy will always be lost from the star (except in a supernova).